Computation of Generalized Nash Equilibria with the GNE package

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OUTLINE

What is a Nash equilibrium (NE)?

Some history,

- An equilibrium concept was first introduced by Cournot in 1838.
- But mathematical concepts had widely spread with *The theory of games and economic behavior* of von Neumann & Morgenstern book in 1947.
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Various names for various games

- Which type of players?
  Cooperative vs. non-cooperative
- Which type of for the strategy set?
  Finite vs. infinite games
  e.g. $\{squeal, silent\}$ for Prisoner's dilemna vs. $[0, 1]$ cake cutting game
- Is there any time involved?
  Static vs. dynamic games
  e.g. battle of the sexes vs. pursuit-evasion game
- Is it stochastic?
  Deterministic payoff vs. random payoff

NON-COOPERATIVE GAMES

Idea : to model competition among players

- Characterization by a pair of
  - a set of stategies S for the n players
  - a payoff function f such that $f_i(s)$ is the payoff for player i given the overall strategy s

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- Nash equilibrium is defined as a strategy in which no player can improve unilateraly his payoff.
- Duopoly example:
  Consider two profit-seeking firms sell an identical product on the same market. Each firm will choose its production rates.
  Let $x_i \in \mathbb{R}_+$ be the production of firm i and d, λ and ρ be constants. The market price is

$$p(x) = d - \rho(x_1 + x_2),$$

from which we deduce the ith firm profit

$$p(x)x_i - \lambda x_i.$$

For $d = 20$, $\lambda = 4$, $\rho = 1$, we get $x^\star = (16/3, 16/3)$.

Extension to generalized NE (GNE)

Definition (NEP)

The Nash equilibrium problem NEP(N, θ_ν, X) for N players consists in finding $x^\star$ solving N sub problems

$$x_\nu^\star \ \textit{solves} \ \min_{y_\nu \in X_\nu} \ \theta_\nu(y_\nu, x_{-\nu}^\star), \forall \nu = 1, \dots, N,$$

where X_ν is the action space of player ν.

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GNE was first introduced by Debreu, [Deb52].

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$$x^\star_\nu \ \textit{solves} \ \min_{y_\nu} \ \theta_\nu(y_\nu, x^\star_{-\nu}) \ \textit{such that} \ x^\star_\nu \in X_\nu(x^\star_{-\nu}), \forall \nu = 1, \dots, N,$$

where $X_\nu(x_{-\nu})$ is the action space of player ν given others player actions $x_{-\nu}$.

Remark: we study a jointly convex case $X = \{x, g(x) \leq 0, \forall \nu, h_\nu(x_\nu) \leq 0\}$.

OUTLINE

Reformulation as a gap

Definition (GP)

Let ψ be a gap function. We define a merit function as

$$\hat{V}(x) = \sup_{y \in X(x)} \psi(x, y).$$

From [vHK09], the gap problem GP(N, X, ψ) is

$$x^\star \in X(x^\star) \textit{ and } \hat{V}(x^\star) = 0.$$

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Example: the Nikaido-Isoda function

$$\psi(x, y) = \sum_{\nu=1}^{N} [\theta(x_\nu, x_{-\nu}) - \theta(y_\nu, x_{-\nu})] - \frac{\alpha}{2} ||x - y||^2.$$

THE minGap FUNCTION

The gap function minimization consists in minimizing a gap function $\min V(x)$. The function `minGap` provides two optimization methods.

- Barzilai-Borwein method

$$x_{n+1} = x_n + t_n d_n,$$

where direction is $d_n = -\nabla V(x_n)$ and stepsize

$$t_n = \frac{d_{n-1}^T d_{n-1}}{d_{n-1}^T (\nabla V(x_n) - \nabla V(x_{n-1}))}.$$

- BFGS method

$$x_{n+1} = x_n + t_n H_n^{-1} d_n,$$

where H_n approximates the Hessian by a symmetric rank two update of the type $H_{n+1} = H_n + auu^T + bvv^T$ and t_n is line searched.

Duopoly example

Let $x_i \in \mathbb{R}_+$ be the production of firm i and $d = 20$, $\lambda = 4$ and $\rho = 1$ be constants. The ith firm profit

$$\theta_i(x) = (d - \rho(x_1 + x_2))x_i - \lambda x_i.$$

```
> minGap(c(10,20), V, GV, method="BB", ...)
**** k 0
 x_k 10 20
**** k 1
 x_k 9.947252 19.50739
**** k 2
 x_k 8.42043 7.216102
**** k 3
 x_k 6.53553 6.066531
**** k 4
 x_k 5.333327 5.333322
$par
[1] 5.333327 5.333322

$outer.counts
    Vhat gradVhat
       5        9

$outer.iter
[1] 3

$code
[1] 0
>
```

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Outer iterations	3
Inner iterations - V	141
Inner iterations - ∇V	269

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Outer iterations	3
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Function calls - V	5
leading to calls - ψ	978
leading to calls - $\nabla\psi$	276
Function calls - ∇V	9
leading to calls - ψ	2760
leading to calls - $\nabla\psi$	652

OUTLINE

1 GAP FUNCTION MINIZATION APPROACH

2 FIXED-POINT APPROACH

3 KKT REFORMULATION

Reformulation of the GNE problem as a fixed-point

Definition (regularized NIF)

The regularized Nikaido-Isoda function is

$$NIF_\alpha(x,y) = \sum_{\nu=1}^{N} [\theta(x_\nu, x_{-\nu}) - \theta(y_\nu, x_{-\nu})] - \frac{\alpha}{2}||x-y||^2,$$

with a regularization parameter $\alpha > 0$*.*

Let $y_\alpha : \mathbb{R}^n \mapsto \mathbb{R}^n$ be the equilibrium "response" defined as

$$y_\alpha(x) = (y_\alpha^1(x), \dots, y_\alpha^N(x)),$$

where

$$y_\alpha^i(x) = \underset{y_i \in X_i(x_{-i})}{\arg\max}\ \mathrm{NIF}_\alpha(x,y).$$

THE FIXEDPOINT FUNCTION

Fixed-point methods for $f(x) = 0$

- "pure" fixed point method

$$x_{n+1} = f(x_n).$$

- Polynomial methods
  - Relaxation algorithm

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n) x_n,$$

    where $(\lambda_n)_n$ is either constant, decreasing or line-searched.
  - RRE and MPE method

$$x_{n+1} = x_n + t_n (f(x_n) - x_n)$$

    where $r_n = f(x_n) - x_n$ and $v_n = f(f(x_n)) - 2f(x_n) + x_n$, with t_n equals to $< v_n, r_n > / < v_n, v_n >$ for RRE1, $< r_n, r_n > / < v_n, r_n >$ for MPE1.
  - Squaring method
    It consists in applying twice a cycle step to get the next iterate

$$x_{n+1} = x_n - 2t_n r_n + t_n^2 v_n,$$

    given t_n such as RRE and MPE.
- Epsilon algorithms: not implemented

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	ψ	$\nabla\psi$	Outer iter
Gap - BB	317	90	2

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SqRRE FP	491	197	4
SqMPE FP	530	166	3

Outline

1 Gap function minization approach

2 Fixed-point approach

3 KKT reformulation

KKT SYSTEM REFORMULATION

DEFINITION (KKT)

From [FFP09], the first-order necessary conditions for ν subproblem states, there exists a Lagrangian multiplier $\lambda_\nu \in \mathbb{R}^m$ such that

$$\begin{aligned} \tilde{L}(x,\lambda) = \nabla_{x_\nu}\theta_{x_\nu}(x) + \sum_{1\leq j\leq m} \lambda_{\nu j}\nabla_{x_\nu} g_j(x) = 0 & \qquad (\in \mathbb{R}^{n_\nu}) \\ 0 \leq \lambda_\nu \perp g(x) \geq 0 & \qquad (\in \mathbb{R}^m) \end{aligned}.$$

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PROPOSITION

The extended KKT system can be reformulated as a system of equations using complementarity functions. Let ϕ be a complementarity function.

$$F_\phi(x,\lambda) = 0 \;\; where \;\; F_\phi(x,\lambda) = \begin{pmatrix} \tilde{L}(x,\lambda) \\ \phi_.(-g(x),\lambda) \end{pmatrix},$$

where $\phi_.$ is the component-wise version of ϕ.

Example: $\phi(a,b) = \min(a,b)$.

THE NEWTONKKT FUNCTION

The problem is

$$\Phi(z) = 0,$$

with $z = (x^T \lambda^T)^T$. It is solved by an iterative scheme $z_{n+1} = z_n + d_n$, where the direction d_n is computed in two different ways:

- Newton method: The direction solves the system

$$V_n d = -\Phi(x_n),$$

  with $V_n \in \partial\Phi(x_n)$.
- The Levenberg-Marquardt method: The direction solves the system

$$(V_n^T V_n + \lambda_k I)d = -V_n^T \Phi(x_n),$$

  where I denotes the identity matrix, λ_k is the LM parameter, e.g. $\lambda_n = || \Phi(z_n)||^2$.

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> NewtonKKT(rep(0, 4), "Leven", ...)
$par
[1]  5.333  5.333 -3e-08 -3e-08

$value
[1] 4.502713e-08

$counts
   phi jacphi
    12     12

$iter
[1] 11
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CONCLUSION

The **GNE** package[1] provides base tools to compute generalized Nash equilibria for static infinite noncooperative games.

Three methods :

- Gap function minimization: `minGap`,
- Fixed-point methods: `fixedpoint`,
- KKT reformulation: `NewtonKKT`.

[1] hosted on R-forge.

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Future developments:

- extends to non jointly convex case for FP and GP methods,
- develop further the KKT reformulation,
- later, extends to finite noncooperative games, e.g. Lemke-Howson algorithm,
- far later, extends to cooperative games!

[1] hosted on R-forge.

REFERENCES

- Gerard Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. U.S.A. (1952).
- Francisco Facchinei, Andreas Fischer, and Veronica Piccialli, *Generalized Nash equilibrium problems and Newton methods*, Math. Program., Ser. B **117** (2009), 163–194.
- Anna von Heusinger and Christian Kanzow, *Optimization reformulations of the generalized Nash equilibrium problem using the Nikaido-Isoda type functions*, Computational Optimization and Applications **43** (2009), no. 3.

Thank you for your attention!