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Iterated function system and simulation of Brownian motion

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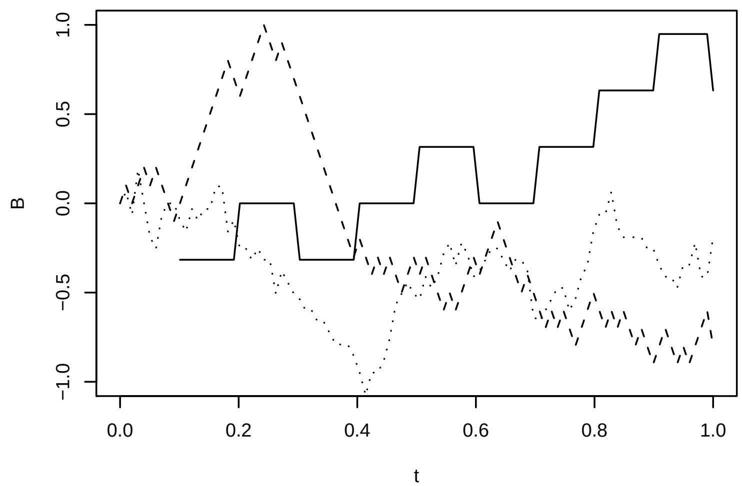
Different ways of simulating BM paths

- simulating increments $B(t) - B(s) \sim N(0, t-s)$
- limit of the random walk $S_n = \sum X_i$, with $P(X_i = \pm 1) = 1/2$

$$\left(\frac{S_{[nt]}}{\sqrt{n}}, t \geq 0\right) \xrightarrow{d} (B(t), t \geq 0)$$

These implies simulation on a grid and between grid points BM path is linearly interpolated



continuous line $n = 10$, dashed line $n = 100$, dotted line $n = 1000$.

Pathwise approximations

Karhunen-Loève / Kac-Siegert decomposition

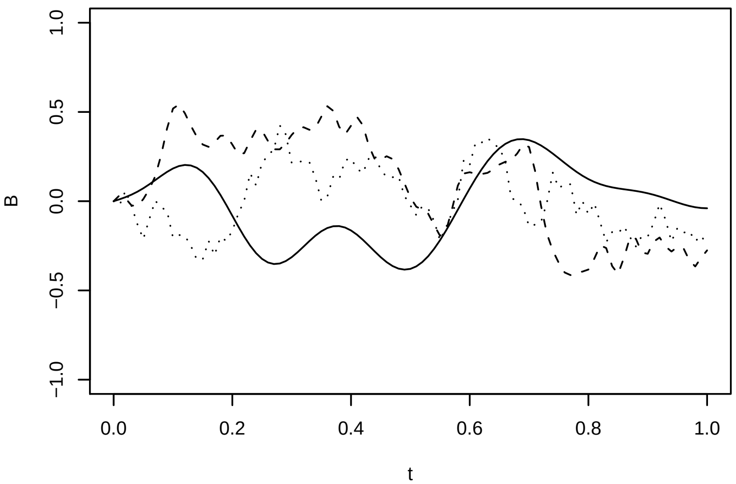
$$B(t, \omega) = \sum_{i=0}^{\infty} Z_i \phi_i(t), \quad 0 \leq t \leq T$$

with

$$\phi_i(t) = \frac{2\sqrt{2T}}{(2i+1)\pi} \sin\left(\frac{(2i+1)\pi t}{2T}\right)$$

ϕ_i a basis of orthogonal functions and Z_i i.i.d. $N(0,1)$

This approximation might be *too* smooth



$n = 10$ (continuous line), $n = 50$ (dashed line) and $n = 100$ terms

IFS-M operator

The IFS-M operator is contractive operatore defined as

$$T(g(x)) = \sum_{k=1}^{N} \left\{ \alpha_k \cdot g\left(\frac{x - a_k}{s_k}\right) + \beta_k \right\}$$

where (α_k, β_k, a_k) can be determined as the solution of a contrained Quadratic Problem given some choice of (a_k, s_k)'s

$$\Delta^2 = ||g - Tg||_2^2 = \min_{\alpha,\beta}$$

under the constraint

$$\sum_{k=1}^{N} c_k(\alpha_k \|g\|_1 + \beta_k) \le \|g\|_1$$

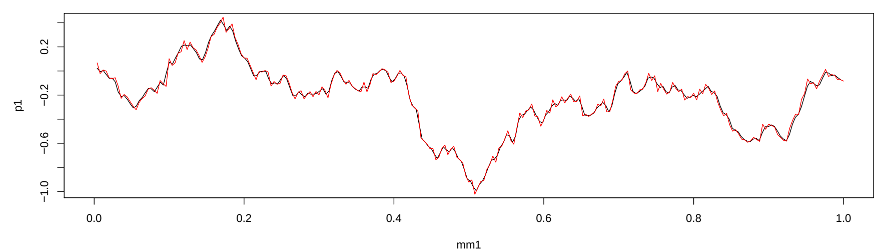
Δ^2 can be rewritten as a quadratic form

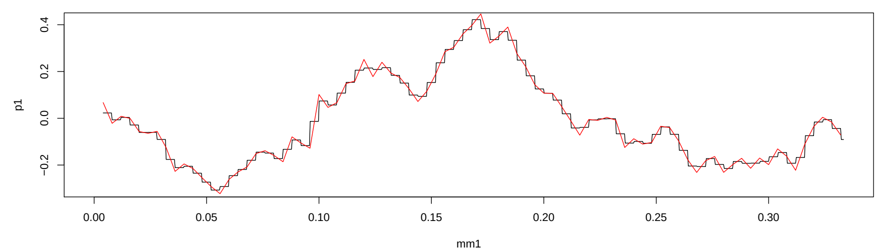
$$\Delta^2 = x^T A x + b^T x + c$$

where $x = (\alpha_1, \ldots \alpha_k, \beta_1, \ldots, \beta_k)$. If $g = BM$ then

- $a_{i,i} = c \int_0^1 B^2(t)dt$
- $a_{N+i,N+i} = s_i$
- $a_{i,N+i} = c \int_0^1 B(t)dt$
- $b_i = -2 \int_0^1 B(t)B((t - a_i)/s_i)dt$
- $b_{N+i} = -2 \int_{a_i}^{a_i+s_i} B(t)dt$

with $c = \int_0^1 |B(t)|dt$





> **Theorem (Self-affine trajectories)**
>
> *Let* (α_k, β_k) *be the solution of* $\Delta^2 = \min_{\alpha.\beta}$ *then the fixed point* $\tilde{B}(t)$ *of the operator* T *satisfies the self affine property*
>
> $$\tilde{B}(w_i(t+h)) - \tilde{B}(w_i(t)) = \alpha_i(\tilde{B}(t+h) - \tilde{B}(t))$$
>
> *where* $w_i(x) = a_i x + s_i$

Which means that the trajectory is made of rescaled copies of itself and here comes the fractal nature of the approximation.

The IFS package

IFS's can be built on distribution functions as well (DSC 2003) and the `ifs` package include both families of operators (IFS-p and IFS-M)

References

- IFSM representation of Brownian motion with applications to simulation, *submitted*.
- A comparative simulation study on the IFS distribution function estimator, *Nonlinear Analysis - Real World Applications*, **6**, 5, 858-873 (2005).
- Approximating distribution functions by iterated function systems, *Journal of Applied Mathematics and Decision Sciences*, **9**, 1, 33-46 (2005).