BradleyTerry2: Flexible Models for Paired Comparisons

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Pair Comparisons

In situations where one object is pitted against another, e.g.

- players/teams in sport
- consumer products in market research
- images in psychology experiments
- plants in pest-resistance trials
- alleles in transmission from parent to child

Bradley-Terry Models

In a **contest** between two **players** i and j, the basic Bradley-Terry Model is given by

$$\text{odds}(i \text{ beats } j) = \frac{\text{pr}(i \text{ beats } j)}{\text{pr}(j \text{ beats } i)} = \frac{\alpha_i/(\alpha_i + \alpha_j)}{\alpha_j/(\alpha_i + \alpha_j)} = \frac{\alpha_i}{\alpha_j}$$

where $\alpha_i, \alpha_j > 0$ are the player **abilities**.

The abilities can be estimated via maximum likelihood by re-framing the model as a logistic model:

$$\text{logit}(\text{pr}(i \text{ beats } j)) = \lambda_i - \lambda_j$$

Structured Bradley-Terry Models

Contest-specific effects

$$\lambda_{ik} = \alpha_i + \sum_r \beta_r x_{ikr}$$

Ability modelled by player attributes

$$\lambda_i = \sum_r \beta_r x_{ir} + e_i$$

The prediction error e_i

- allows for variability between players with equal covariate values
- induces correlation between comparisons with a common player

The *BradleyTerry2* package

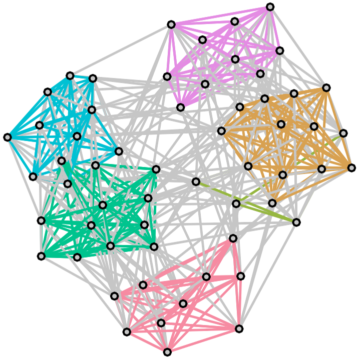
New features to accommodate general Bradley-Terry model

- flexible formula interface to modelling fitting function `BTm()`: allows player-specific, judge-specific, contest-specific variables and random effects
- PQL algorithm for estimation of GLMMs
- efficient data management of multiple data frames

Best of original *BradleyTerry* package

- translation of ability formula to design matrix
- methods for fitted model object, e.g. anova, BTabilities
- handling of missing data in player covariates

College Ice Hockey: Men's Division I



1083 games from the 2009-10 composite schedule

58 teams from 6 conferences

Results in data frame `icehockey`

- `visitor` visiting team
- `opponent` usually home team
- `result` 1, 0 or 0.5

KRACH Ratings

Ken's Ratings for American College Hockey are obtained from a standard Bradley-Terry model with a separate ability for each team:

```
> standardBT <- BTm(outcome = result,
+       player1 = data.frame(team = visitor),
+       player2 = data.frame(team = opponent),
+       id = "team", formula = ~ team, data = icehockey)
```

The default behaviour provides some simplification

```
> standardBT <- BTm(outcome = result,
+       player1 = visitor, player2 = opponent,
+       id = "team", data = icehockey)
```

Converting BTm Results to KRACH

The `BTabilities` function returns the log-abilities and s.e.

```
> head(BTabilities(standardBT), 4)
```

```
                     ability      s.e.
Alaska-Anchorage    0.0000000 0.0000000
Air Force          -1.4135091 0.6560509
Alabama-Huntsville -0.6825408 0.6052347
American Int'l     -2.9316119 0.7121359
```

KRACH rescales the abilities so that KRACH $= 100 \Rightarrow$ RRWP $= 0.5$

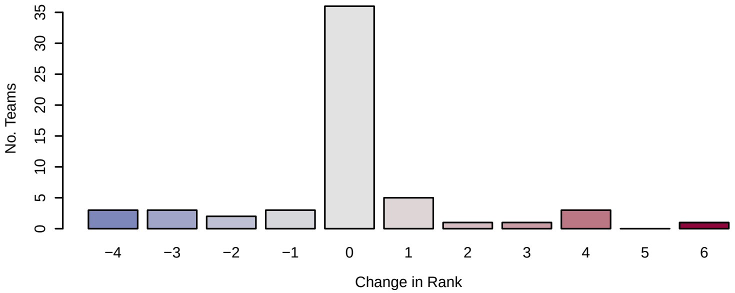
```
> KRACH <- exp(BTabilities(standardBT)[,1])*scale
> head(sort(round(KRACH, 1), decr = TRUE))
```

```
        Denver          Miami       Wisconsin
         543.0          488.2           481.3
  North Dakota Boston College St. Cloud State
         434.3          346.2           345.3
```

Home Ice Advantage

The official NCAA ranking gives more credit for neutral-site/road wins. Equivalently we can adjust for home ice advantage

```
> levelBT <- BTm(result,
+     data.frame(team = visitor, home.ice = 0),
+     data.frame(team = opponent, home.ice = home.ice),
+     ~ team + home.ice, id = "team", data = icehockey)
```



Effect on Selection?

The 6 regional winners automatically qualify for the NCAA tournament, whilst another 10 are selected by ranking

KRACH	level KRACH	NCAA
Denver	Denver	Miami
Miami	Miami	Denver
Wisconsin	Wisconsin	Wisconsin
St. Cloud State	St. Cloud State	St. Cloud State
Minnesota Duluth	Bemidji State	Bemidji State
Northern Michigan	Northern Michigan	Yale
Colorado College	New Hampshire	Northern Michigan
New Hampshire	Minnesota Duluth	New Hampshire
Minnesota	Colorado College	Alaska
Bemidji State	Vermont	Vermont

Lizards Data Revisited

Data Structure in R

```
> str(flatlizards)
List of 2
 $ contests  :'data.frame': 100 obs. of  2 variables:
  ..$ winner: Factor w/ 77 levels "lizard003","lizard005",..: 27 33
  ..$ loser : Factor w/ 77 levels "lizard003","lizard005",..: 3 6 7
 $ predictors:'data.frame': 77 obs. of  18 variables:
  ..$ id          : Factor w/ 77 levels "3","5","6","9",..: 1 2 3 4
  ..$ throat.PC1  : num [1:77] -1.16 -13.19 -12.47 4.75 -13.47 ...
  ..$ throat.PC2  : num [1:77] 1.066 2.127 -0.771 8.399 -1.968 ...
  ..$ throat.PC3  : num [1:77] 3.2152 0.8776 -1.6139 0.0786 0.4982
  ...
```

Model in Whiting et al, *Animal Behaviour*, 2006

```
> liz <- BTm(1, winner, loser, ~throat.PC1[..] + throat.PC3[..] +
    head.length[..] + SVL[..] + (1|..), data = flatlizards)
```

	No random effects		Random effects	
	Estimate	Std. Error	Estimate	Std. Error
lizard096	16.42		36.68	
lizard099	0.84	1.16	0.95	1.28
throat.PC1	−0.09	0.03	−0.09	0.04
throat.PC3	0.34	0.11	0.37	0.15
head.length	−1.13	0.49	−1.38	0.74
SVL	0.19	0.10	0.17	0.14
σ_e			1.1	0.3

Main conclusions from original study:

- Overall brightness (PC1) and UV intensity (PC3) of the throat are clearly significant predictors of fight-winning ability.
- PC3 has by far the largest effect: in a contest between lizards at ± 2 standard deviations the odds are estimated as ≈ 30 in favour of the lizard with greater UV reflectance on the throat.

Thankfully unaffected by allowing for variation between lizards with the same covariate values!

Summary

`BradleyTerry2` can be downloaded from
`http://cran.r-project.org/package=BradleyTerry2`

Package vignette gives further examples.

Further development planned, e.g.

- better handling of ties
- random effects for judges