Estimation and Testing of Portfolio Value-at-Risk Based on L-Comoment Matrices

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Theme

- This study employs L-comoments introduced by Serfling and Xiao (2007) into portfolio Value-at-Risk estimation through two models: the Cornish-Fisher expansion (Draper and Tierney 1973) and modified VaR (Zangari 1996).
- Backtesting outcomes indicate thatmodified VaR outperforms and L-comoments give better estimates of portfolio skewness and excess kurtosis than do classical central moments in modeling heavy-tailed distributions.

PVaR

For a portfolio with n assets, PVAR at a confidence level α is specified as follows:

$$VaR(\alpha) = -F^{-1}(\alpha). \qquad (1)$$

Here, the return series and their respective weights are denoted as $r = (r_1, \ldots, r_n)'$ and $\omega = (\omega_1, \ldots, \omega_n)'$, while $F^{-1}(\bullet)$ is the quantile function associated to the cumulative density function $F(\bullet)$ of the portfolio return distribution (r_p).

PVaR

Under a location-scale representation, the portfolio return can be expressed as:

$$r_p = \omega' \mu + \sqrt{m_2} u, \qquad (2)$$

where μ and m_2 represent the portfolio mean and the second central moment.

Here, u denotes a random variable with distribution function $G(\cdot)$ of zero mean and unit variance.

GVaR

Gaussian VaR, GVaR, PVaR under multivariate normality assumption, can be expressed as:

$$GVaR(\alpha) = -\omega'\mu - \sqrt{m_2}\,\Phi^{-1}(\alpha), \qquad (3)$$

where $\Phi^{-1}(\alpha)$ denotes the quantile function at α significance level of standard normal distribution.

refinements

- major ones such as:
  - Draper and Tierney (1973) extend more terms to enhance estimation accuracy
  - Zangari (1996) corrects the skewness and excess kurtosis of the Gaussian quantile function and proposes the modified VaR (mVaR).
- the performances of those two refinements are contingent upon an estimation of the moments.
- However, current practices mostly rely on the traditional moment estimation.

The Cornish-Fisher expansion

- known for its decomposable and analytical expression, because of a normality assumption on the return distribution
- However, the adjustment factor is reliable only if the distribution is close enough to being normal

CFVaR

$$CFVaR = -\omega'\mu - \sqrt{m_2}G^{-1}(\alpha)$$

$$G^{-1}(\alpha) = z_\alpha + \tfrac{1}{6}(z_\alpha^2 - 1)s_p + \tfrac{1}{24}(z_\alpha^3 - 3z)k_p - \tfrac{1}{36}(2z_\alpha^3 - 5z_\alpha)s_p^2$$

It is proved that the Cornish-Fisher approximations hardly improve performance even when we increase the order of approximation (see, for example, Hardle, Kleinow, and Ulfig (2002) and Jasche (2002)). Accordingly, this study only extends the mVaR and CFVaR expressions to the second order.

mVaR by Zangari (1996)

$$mVaR(\alpha) = GVaR(\alpha) + \sqrt{m_2}\left[-\tfrac{1}{6}\left(z_\alpha^2 - 1\right)s_p - \tfrac{1}{24}\left(z_\alpha^3 - 3z\right)k_p + \tfrac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)s_p^2\right]$$

where s_p and k_p are the portfolio skewness and excess kurtosis, respectively,

z_α equals $\Phi^{-1}(\alpha)$

corrects the skewness and excess kurtosis of GVaR

mVaR

- its calculation relies on the first four moments.
- Favre and Galeano (2002) conclude that the skewness and the kurtosis effect are high if the VaR is computed at 99%.
- It is expected that the moment estimation plays an important role in approximating the downside risk at extremal significance levels.

moment estimation

- a crucial role in financial analysis - e.g. portfolio optimization and capital asset pricing model
- yet it is criticized for a heavy reliance on moment assumptions of second order or higher in the multivariate portfolio analysis
- The assumptions for moment estimation are hardly supported by financial return series
  - the traditional central moments are confined to sufficiently light-tailed distributions, while financial return series exhibit heavy-tailed properties

Central Moments

Traditionally, the qth orders of portfolio central moments are defined as $m_q = E\left[\left(r_p - \omega'\mu\right)^q\right]$, and we have:

$$m_2 = \omega'\Sigma\omega$$

$$m_3 = \omega' M_3 \omega\left(\omega \otimes \omega\right)$$

$$m_4 = \omega' M_4 \omega\left(\omega \otimes \omega \otimes \omega\right),$$

where $\otimes$ stands for the Kronecker product.

$$M_3 = E\left[(r-\mu)(r-\mu)' \otimes (r-\mu)\right] \quad M_4 = E\left[(r-\mu)(r-\mu)' \otimes (r-\mu)'(r-\mu)'\right]$$

Portfolio skewness & excess kurtosis

The portfolio skewness (s_p) and excess kurtosis (k_p) are given by:

$$s_p = \frac{m_3}{(m_2)^{3/2}}$$

$$k_p = \frac{m_4}{(m_2)^2} - 3$$

L-moments proposed by Hosking (1990)

- a better alternative for higher moment estimators,
- based solely on a finite first moment assumption
- analogous to central moments and give a coherent estimation with traditional central moments
- give a better description of heavy-tailed distributions that financial return series usually demonstrate
- Their application can be exercised not only parametrically, but also in a semiparametric and non-parametric modeling setting.

multivariate L-moments or L-comoments by Serfling and Xiao (2007)

- Extension of L-moments to a multivariate scenario
- i.e. Gini-covariance, L-coskewness, and L-cokurtosis for orders of 2, 3, and 4, respectively.
- While analogous to traditional central moments, L-comoments are effective new descriptive tools and outperform in a non-parametric moment-based description of a possibly heavy-tailed distribution.
- So far, L-comoments have not been applied to PVaR estimation and the estimation performance still waits to be evaluated via backtesting.

L-comoments

For the n-ordered observations from a univariate distribution $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$,

the nth L-moment is defined as:

$$\lambda_n = n^{-1} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} E\left(x_{n-j:n}\right) \quad (8)$$

L-moments possess attractive properties in comparison to classical central moment analogues, including finite if the first moment is finite and the estimates show unbiasedness.

L-comoments

The L-moments sequence (λ_n) can also be expressed as the expected value of an order statistics, i.e.:

$$\lambda_n = \int_0^1 F^{-1}(u) P^*_{n-1}(v) dv, \qquad (9)$$

where $P^*_n(v) = \sum_{j=0}^{n} p^*_{n,j} v^j$ with $p^*_{n,j} = (-1)^{n-j} \binom{n}{j} \binom{n+j}{j}$

By the orthogonality of orthogonal polynomials $p^*_{n,j}$

λ_n captures the information about F

L-comoments

covariance, $\lambda_n = Cov\left[x, P^*_{n-1}\left(F\left(x\right)\right)\right]$. (10)

Recall that the qth order central comoment matrices are defined as:

$$Cov\left[x^i - \mu_i, \left(x^j - \mu_j\right)^{q-1}\right]. \quad (11)$$

Thus, the qth order L-comoment can be defined as:

$$\lambda_{q[ij]} = Cov\left[x^1, P^*_{q-1}\left(F_j\left(x^j\right)\right)\right], \quad q \geq 2, \quad (12)$$

if Equations (10) and (11) are combined together.

L-comoments

- based on a comprehensive pairwise approach for descriptive measures with dimensions higher than 2.
- developed toward dispersion, correlation, skewness, and kurtosis, etc. in a multivariate setting.

Backtesting

- two major criteria for backtesting:
  - unconditional rate of exceedances (UC)
  - $$LR_{UC} = -2\ln\left[(1-p)^{T-N}\, p^{N}\right] + 2\ln\left\{\left[1-N/T\right]^{T-N}(N/T)^{N}\right\} \sim \chi^2(1), \qquad (13)$$
  - independence of the exceedances (IND).
  - $$LR_{IND} = -2\ln\left[(1-\pi)^{(T_{00}+T_{10})}\,\pi^{(T_{01}+T_{11})}\right] + 2\ln\left[(1-\pi_0)^{T_{00}}\,\pi_0^{T_{01}}\,(1-\pi_1)^{T_{10}}\,\pi_1^{T_{11}}\right] \sim \chi^2(1), \qquad (14)$$

$$LR_{CC} = LR_{UC} + LR_{IND}$$

Table 2: Summary Statistics

	CAD	AUD	IDR	THB	KRW	GBP
Min	-3.74E-02	-6.80E-02	-3.32E-01	-6.32E-02	-2.03E-01	-4.69E-02
Mean	6.61E-07	6.93E-06	4.39E-04	8.50E-05	1.57E-04	2.70E-05
Median	0.00E+00	-7.70E-05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Max	3.31E-02	7.61E-02	3.03E-01	7.40E-02	1.35E-01	3.96E-02
Total N	4.03E+03	4.03E+03	4.03E+03	4.03E+03	4.03E+03	4.03E+03
Std Dev.	4.75E-03	7.76E-03	1.75E-02	5.83E-03	9.56E-03	5.81E-03
Skewness	7.06E-02	6.15E-01	5.74E-01	6.05E-01	-1.16E+00	2.85E-01
Kurtosis	5.99E+00	1.06E+01	9.96E+01	3.72E+01	1.08E+02	5.17E+00
Jarque-Bera normality test statistics	6010.64 (0.00)	19210.48 (0.00)	1661269 (0)	231205.3 (0.0)	1943319 (0.00)	4518.802 (0.000)

Table 3: Estimates of Portfolio Skewness and Excess Kurtosis

Method		Classical Central Moments	L-Comoments
Portfolio I: CAD+AUD	s_p	0.39341181	0.26537
	k_p	4.881089818	26.0841
Portfolio III: IDR+THB	s_p	0.574206378	0.58633
	k_p	80.48015391	47.9926
Portfolio II: KRW+GBP	s_p	-0.65107669	0.64693
	k_p	56.47661884	48.5317

Note:

s_p : portfolio skewness, k_p : portfolio excess kurtosis

Table 4: Outcomes of Backtesting of PVAR Estimates

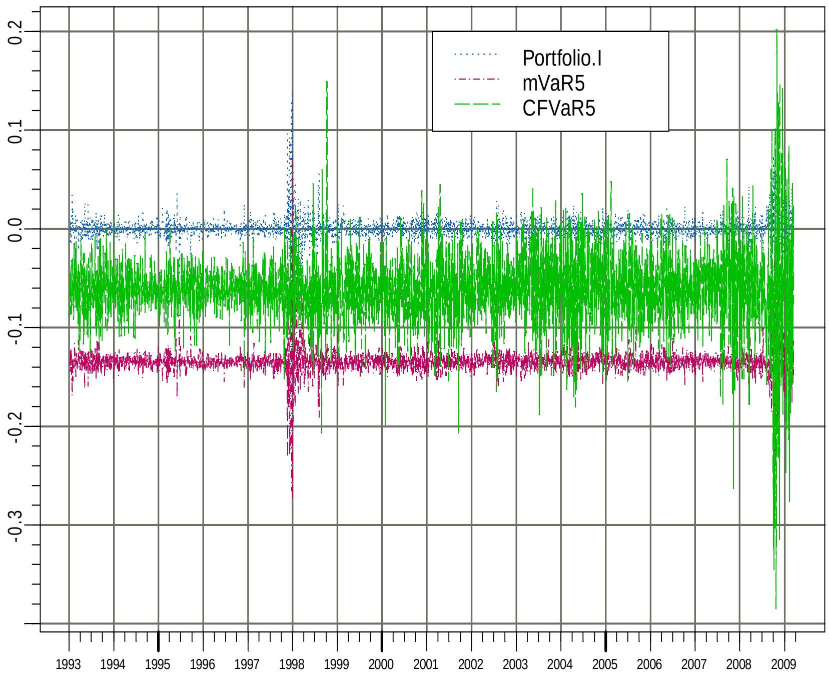
Model		mVaR(M)		mVaR(L)		CFVaR(M)		CFVaR(L)	
Significance level		1%	5%	1%	5%	1%	5%	1%	5%
Portfolio I: CAD+AUD	UC	×	×	×	×	×	×	×	×
	IND	○	○	○	○	○	○	○	○
	CC	×	×	○	○	×	×	×	○
Portfolio II: IDR+THB	UC	×	×	×	×	○	×	×	×
	IND	×	×	○	○	×	×	×	×
	CC	×	×	○	×	×	×	×	×
Portfolio III: KRW+GBP	UC	×	×	×	×	×	×	×	×
	IND	×	×	○	○	×	×	×	×
	CC	×	×	○	×	×	×	×	×

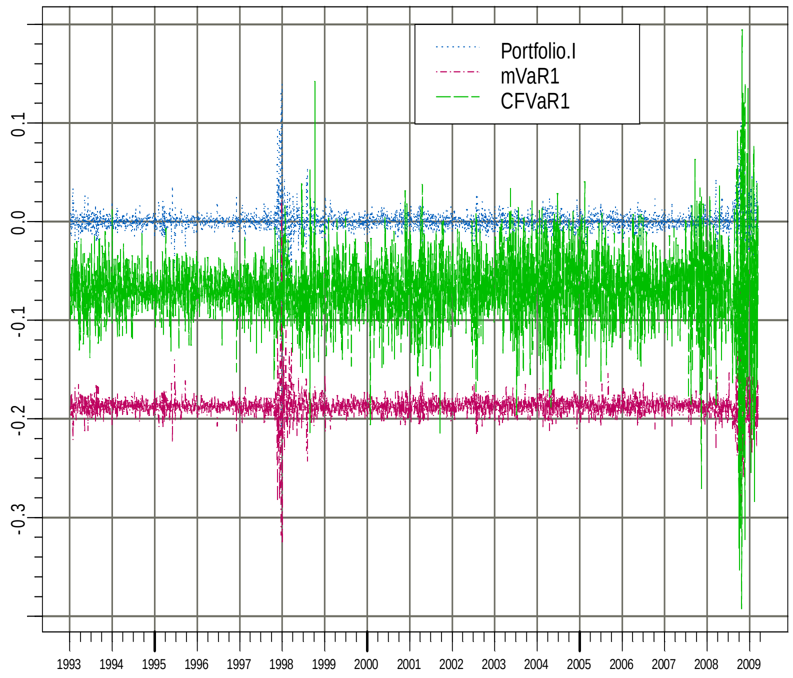
Note:

1. × and ○ represent rejection and non-rejection of the null hypothesis, respectively. The significance level for each null hypothesis is set at 10%.

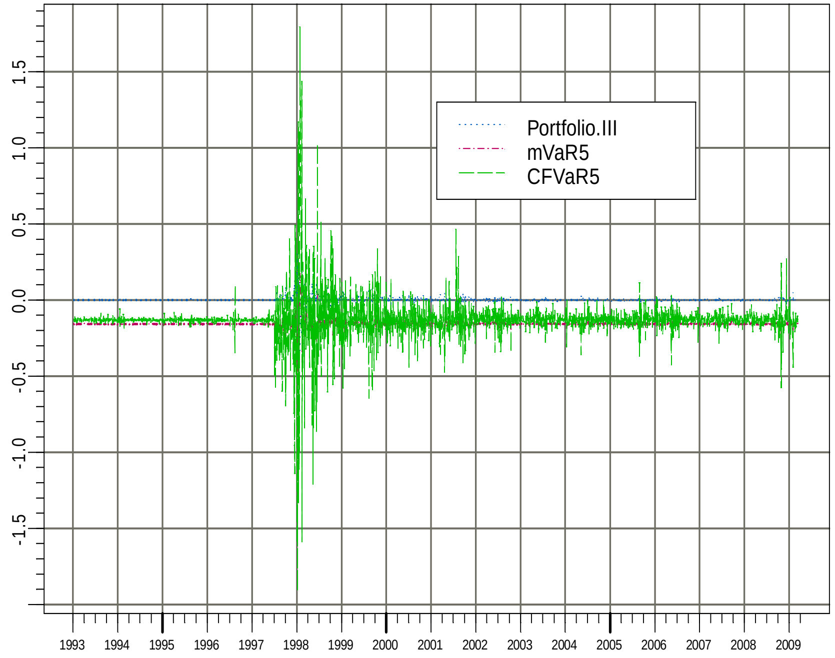
2. UC: unconditional coverage test; IND: independence test of the exceedances; CC: conditional coverage test

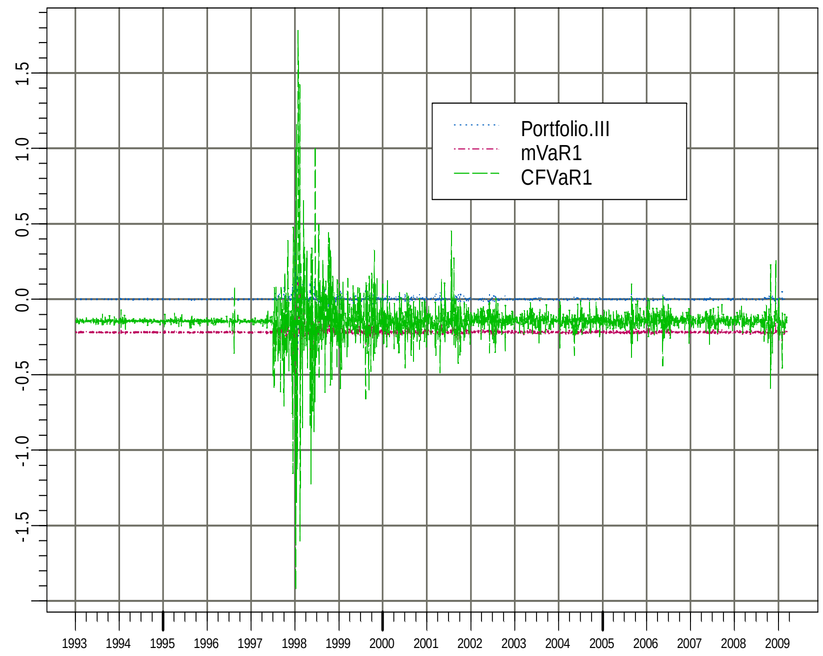
Portfolio Returns vs. Their PVaR Estimates
Portfolio I: CAD+AUD



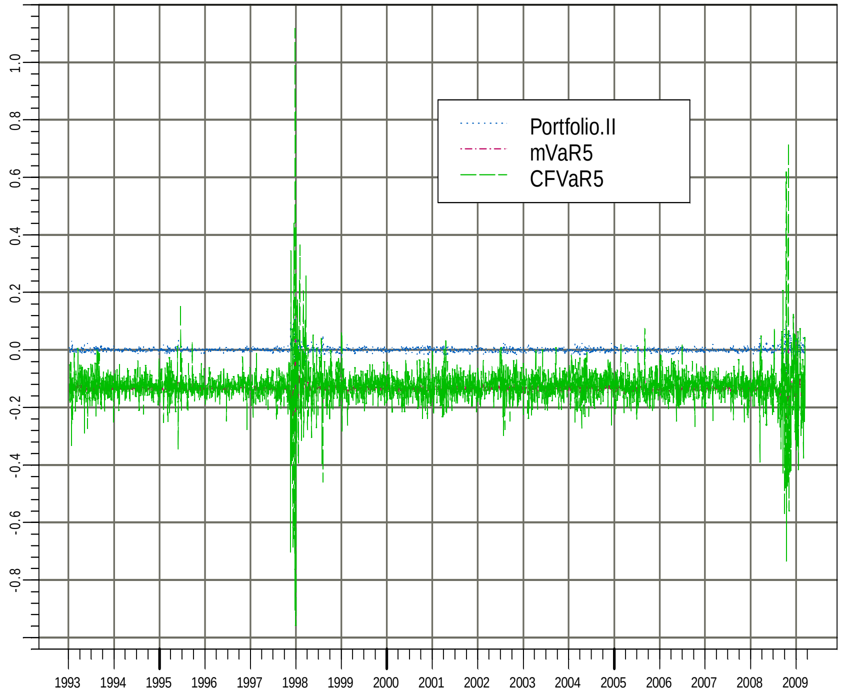


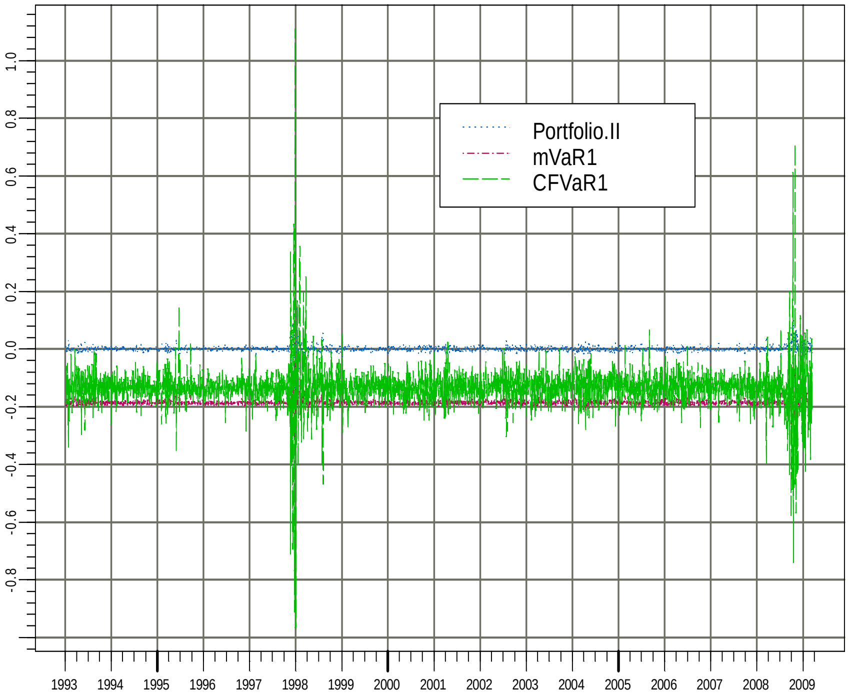
Portfolio Returns vs. Their PVaR Estimates
Portfolio II: IDR+THB





Portfolio Returns vs. Their PVaR Estimates
Portfolio III: KRW+GBP





conclusions

- Cornish-Fisher expansion and mVaR are the major attempts
  - based on the assumptions that an adjustment in the higher moments or correction of portfolio skewness and excess kurtosis can help improve the estimation
- this study highlights the estimation issues of the key components: the central moments

conclusions

- Cornish-Fisher expansion is not suitable for the downside risk estimation of multivariate non-normal returns.
- mVaRs give better performances at the 1 and 5% significance levels
- L-comoments enhance the outperformance, and backtesting offers favorable outcomes
- classical central moments may not be suitable for heavy-tailed distribution in estimating portfolio skewness and excess kurtosis